

# UNCLASSIFIED

|  |
|--|
|  |
|  |
|  |
|  |
| AD NUMBER  |
| AD866692   |
| NEW LIMITATION CHANGE  |
| TO<br>Approved for public release, distribution unlimited  |
| FROM<br>Distribution authorized to DoD only;<br>Administrative/Operational Use; 15 NOV 1969. Other requests shall be referred to Naval Ordnance Lab., White Oak, MD. |
| AUTHORITY  |
| USNOL ltr dtd 12 Dec 1972  |

THIS PAGE IS UNCLASSIFIED

ORD - 9132

NOLTR 69-204

C.1

TR-69-204

669928-AD-866692  
Dept. of Navy  
Washington 25, D. C. 20384

RETURN TO  
TECHNICAL LIBRARY FRANCH  
NAVAL ORDNANCE SYSTEMS DIVISION

Do not forward this copy to other agencies  
without authorization of NOLTR (ORD-0150)

APPLICATION OF LANCHESTER ANALYSIS TO  
A MINING CAMPAIGN

By  
Thomas E. Phipps, Jr.

NOL

15 NOVEMBER 1969

UNITED STATES NAVAL ORDNANCE LABORATORY, WHITE OAK, MARYLAND

NOLTR 69-204

ATTENTION

Each transmittal of this document out-  
side the Department of Defense must  
have prior approval of NOL.

APPLICATION OF LANCHESTER ANALYSIS TO A MINING CAMPAIGN

Prepared by:  
Thomas E. Phipps, Jr.

ABSTRACT: Lanchester-type equations are derived for a mine warfare campaign in which one opponent seeks to blockade the ports of another. The effects of mutual attrition of minelaying units, counter-minelaying forces, minesweepers, and target traffic (merchant shipping or submarines) are treated and conditions for victory or stalemate are derived. Only steady-state solutions of the equations are examined. These appear adequate to treat a broad range of questions about the campaign, including the effect of mine design and tactics on total investment levels needed to win. The particular advantage of this type of model is that it provides an easy, though crude, means of estimating the effect of even the most minor design and situational variables on the overall course of the campaign. For this reason it is hoped that it can prove useful to the mine designer or tactician, when supplemented by the necessary auxiliary models for parameter evaluation. The present report develops only the Lanchester model, without attempting realistic applications.

PUBLISHED 15 NOVEMBER 1970

U. S. NAVAL ORDNANCE LABORATORY  
WHITE OAK, MARYLAND

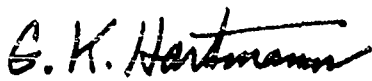
NOLTR 69-204

15 November 1969

## APPLICATION OF LANCHESTER ANALYSIS TO A MINING CAMPAIGN

In the aftermath of the MINETECH Report (NOLTR 69-1 of 17 Jan 1969, Secret) it became clear that existing minefield theory, as summarized for example in Proceedings of the Conferences on the Naval Minefield (NAVORD Report 6020 et. seq., Secret) was inadequate to treat mine design issues that were affected in an essential way by the attrition of mine delivery units. An example of such an issue is the question of optimum standoff range for terminally self-propelled mines delivered by submarines or by aircraft. The present report is an unclassified description of a mathematical model designed to have the capacity to deal in simplified and generalized terms with questions of this kind. All calculations in the present report are of a purely illustrative nature, intended to show the type of results the model can yield without employing classified input data. Thus the present study prepares the way for further analysis, but does not attempt to answer specific questions of the kind just mentioned. The work reported here was supported under Task No. ORD-531-215/UF 17-351-503.

GEORGE G. BALL  
Captain, USN  
Commander

  
G. K. HARTMANN  
By direction

# CONTENTS

|  | <u>Page</u> |
|--|-------------|
| INTRODUCTION. . . . .  | 1           |
| SCENARIO. . . . .  | 2           |
| MODEL CAPABILITIES AND LIMITATIONS. . . . .                  | 3           |
| EXAMPLE OF RESULTS. . . . .                                  | 5           |
| SUMMARY . . . . .  | 12          |
| APPENDIX A. LANCHESTER'S EQUATIONS FOR MINE WARFARE. . . . . | A-1         |
| APPENDIX B. FIGURES OF MERIT . . . . .                       | B-1         |

# ILLUSTRATIONS

## Title

## Figure

- |   |  |   |
|---|--|---|
| 1 | Stalemate Curves for Case of Greater Spending by the Attacker (Miner). Hypothetical Parameter Values. Defender Maintains Constant Shipping Level (25% Port Utilization). . . . . | 8 |
| 2 | Stalemate Curves for Higher Values of Spending Ration. Hypothetical Parameter Values. Defender Maintains Constant Shipping Level (25% Port Utilization) . . . .                  | 9 |

# REFERENCES

- (a) F. W. Lanchester, Aircraft in Warfare; the Dawn of the Fourth Arm (Constable & Co., London, 1916)
- (b) P. M. Morse and G. E. Kimball, Methods of Operations Research (Wiley, New York, 1951.)

## 1. INTRODUCTION

1. During World War I a British analyst, F. W. Lanchester, successfully described the effects of aerial combat by means of simple first-order partial differential equations similar to those long used by chemists to describe reaction rates. (See references (a) and (b).) Later applications of the method to World War II conflicts (e.g., the Iwo Jima campaign) and more recently to analysis of air-ground combat have shown the breadth of potential applications. In the present report we derive Lanchester's equations for a particular type of mining campaign and give a hypothetical example of the kind of results the equations can yield.

2. In general Lanchester-type analysis is most useful for giving an over-all view of a conflict situation by exhibiting the effects of nonlinear interactions of the various types of mutual attrition that are going on. Because the model is an analytic one, the reasons for its output behavior are usually transparently obvious or easily determined. This is often not the case with more elaborate and "accurate" computer simulations, in which the degree of detail is so great that major causal relationships can be lost from sight. Also, with the analytic approach computer time is seconds rather than hours. Hence, the range of parametric sensitivity investigations can be extended with economic advantage. Where the accuracy alleged for simulations of combat situations is out of balance with the reliability of the parameter data inputs, the initial compromises of accuracy inherent in an analytic approach may be fully warranted. Ideally, the two approaches should complement each other: a "first cut" analysis should always be made, to obtain a bird's-eye view; and this should be followed by a detailed simulation if warranted. In practice available time and effort are seldom adequate to support both. Where a choice must be made, economy of force would in general prescribe an analytic investigation.

## 2. SCENARIO

3. To exemplify a Lanchester analysis of mine warfare we consider the following situation. Two major powers, A and B, confront each other in a war-at-sea context. Opponent A seeks to blockade the ports of B by means of air- or submarine-delivered mines. Opponent B seeks to counter this by two types of measure of the nature of "prevention" and "cure." The prevention force consists of fighter aircraft (or SSK submarines in the case of submarine delivery of mines) with airborne radar support. The "cure" consists of a force of minesweepers. The prevention force exacts a certain toll of the mine-delivery units. The latter, presumed capable of fighting back, in turn exact a toll of B's attrition units. As a result of successful mine deliveries by A, a certain minefield threat level is built up outside B's ports, and this attrites B's "target traffic" -- which may be merchant shipping, if trade strangulation is the object of the blockade, or submarines, if the blockade is an ASW one. It also exacts some toll of the minesweepers. It is assumed that B does not accept blockade passively, but continues to push his target traffic through, taking his losses and replacing them. Although the above refers explicitly to "ports", it will be understood that the model applies as well to straits or other convenient sites of mining blockade. Similarly for minesweeping one can substitute minehunting or other countermeasures activity. The scenario just described should apply satisfactorily to a surface shipping blockade, since in that case the mining campaign can be fairly clearly decoupled from the rest of the war through the assumption that opponent B must maintain some definite minimum shipping level or lose the war. For the ASW version of the scenario, the separation is not so clear, since it may be difficult to identify a threshold number of opponent B's operational submarines below which B can be said to have lost the war. As long as any of B's submarines remain active they can still do damage. Without broader analysis it is hard to assess the implications of that damage. Hence the present model in the ASW context should in principle be embedded in a larger model (which could also be of Lanchester type) describing the attrition of opponent A's merchant shipping by B's submarines that get through the minefields. This has not been attempted; nor has attention been given to non-mine or combination submarine-mine barriers.

### 3. MODEL CAPABILITIES AND LIMITATIONS

4. Lanchester's equations for the above-described campaign are developed in Appendix A. The model is simplified in one important respect: namely, only material procurement costs on the two sides are considered, not maintenance, support, and operating costs. The model could easily be modified to include the latter costs; but they have been deliberately omitted in order to focus attention on the material attritional aspect of the campaign. The viewpoint is adopted that

a. a major factor determining the outcome of a war is attritional losses and the economic strains entailed in making replacements under the constraint of wartime scarcities,

b. once forces have been procured a way to fund their operations "out of another pocket" will always be found.

5. Although the time-dependent Lanchester equations are derived in Appendix A, we shall be concerned in this report only with their steady-state solutions. By examining the conditions for a steady state (stalemate of indefinite duration) we can learn most of what needs to be known about this type of mining campaign, such as who wins or loses in what circumstances and how the outcome is affected by parameter value changes. The only thing that cannot be learned is the time relationships -- how quickly the issue is settled. Since strategies and force level commitments are variable during a real campaign, and not predictable reliably by any theoretical means including simulation, an ability to calculate transient effects would not provide a vastly more credible basis for predicting the course of an actual campaign than would the steady-state analysis, unrealistic as the steady state itself may be. Thus, as a practical matter, we do not lose a great deal by omitting to solve the simultaneous nonlinear differential equations in Appendix A for the case of time-dependent force levels.

6. The model involves close to a dozen general parameters. A few of these, such as unit costs, are conceptually simple and specific. The others, such as an aircraft combat efficiency parameter, each involve a separate operations research problem for their evaluation. Since the



present report is unclassified, no attempt is made in it to deal realistically with these O. R. problems. Instead we simply use arbitrary numbers in an illustrative example, with a stern warning to all beholders not to take the numbers seriously. Some, in fact, have been deliberately chosen to be unrealistic.

7. Given careful attention to the above-mentioned O. R. problems, the model should be capable of answering (in a crude way, and subject to all normal reservations about the relationship between analysis and real-world decisions) a wide range of questions bearing on mine design and tactics, such as:

a. What should a mine cost? What stand-off delivery capabilities of the mine itself are warranted cost-wise to reduce delivery-vehicle attrition?

b. What mix of mines and obstructors is best? What obstructor costs are warranted?

c. Should mines attack sweepers or let them pass? Does it pay to use high ship counts or is more frequent replenishment with low ship-count settings preferable?

d. For the defender, what balance between investments for prevention and cure (say, fighter aircraft vs minesweepers) is desirable? Etc.

8. The answers to such questions yielded by the model will of course be wholly predicated on the assumed scenario and will be inapplicable to campaigns of a radically different nature.

#### 4. EXAMPLE OF RESULTS

9. To illustrate in an unclassified way the kind of results obtainable from the model of Appendix A we introduce the following fictitious parameter values. (The symbols are those used in Appendix A.)

$e^{-1}$  = av. mine-delivery unit (aircraft) procurement cost = 3 \$M.

$g^{-1}$  = av. minesweeper procurement cost = 3 \$M.

$j^{-1}$  = av. target vessel procurement cost = 10 \$M.

$m^{-1}$  = av. anti-minelayer attrition unit (fighter aircraft) procurement cost = 1.5 \$M.

$c_m$  = procurement cost per mine = \$3000.

$n_s$  = av. no. of sorties per mine-delivery aircraft per month  
= 15

$n_m$  = av. no. of mines carried per sortie = 4

$A_m$  = total shipping channel area mined = (600 ft) x (100 n. mi.)

$F$  = av. fraction of delivered mines planted in shipping channels  
= 0.2

$f_1$  = av. fraction of mines in shipping channels operable and on count 1 at any time = 0.05

$f_2$  = av. fraction of mine explosions producing damage equivalent to sinking = 0.75

$w$  = av. actuation width of a mine = 100 ft.

$W_s$  = av. minesweeper swept path width = 300 ft.

$v_s$  = av. minesweeper speed when sweeping = 6 kts.

$d_s$  = av. minesweeper duty cycle (fraction of time spent sweeping) = 0.25

$v_t$  = target vessel av. speed in traversing minefield = 5 kts.

$T_1$  = target vessel average cycle (round-trip) time = 1.5 mos.

$B_c$  = av. swept channel width = 600 ft.

$n_o$  = no. of ports mined = 10.

$f_p$  = port utilization factor (fraction of total capacity in use) = 0.25.

$P$  = total tonnage handling rate of all mined ports = 1,000,000 GWT/mo.

$T_o$  = target class vessel average tonnage = 3500 GWT.

$n_a$  = av. minelaying raid size = 8 aircraft.

$p_{AB}$  = probability per sortie that one of the minelaying aircraft is killed by one of the defender's fighter aircraft = 0.02.

$q$  = mean rate of failure of an average mine = 0.

$P_{BA}$  = probability per sortie that one of the minelaying aircraft kills one of the fighter aircraft that attacks it = 0.005

$n_c$  = av. no. of ship counts actuated per sweeper pass per mine = 2.

$k_1$  = relative vulnerability of av. minesweeper to sinking by mines, compared to av. target vessel vulnerability = 0.1.

10. With these numerical input values we calculate from the formulas of Appendix A

$$\epsilon_0 = 0.005$$

$$R_0 = 0.125$$

$$R_1 = 25.0$$

( $R_1 > R_0$  as required by Equation (22), Appendix A). From Equation (32)' of Appendix A we find

$$B_3 = \text{defender's target vessel force operational} \approx 107 \text{ vessels,}$$

this corresponding to  $f_p = 25\%$  port utilization.

11. If we consider a fixed value of the ratio  $R$  of the defender's total spending rate to the attacker's (miner's) total spending rate, and suppose that the defender maintains his operational shipping at the constant level  $B = 107$  vessels, then Equation (21)", Appendix A, provides a functional relationship between opponent B's defensive aircraft inventory and his minesweeper inventory. This relationship expresses the condition for a steady-state solution of the Lanchester equations -- i.e., the condition for existence of a stalemate.

12. The condition is represented graphically in Figures 1 and 2. The number of minesweepers present in opponent B's steady-state inventory is plotted against his inventory of defensive aircraft. Hence, what we are showing is "cure" vs. "prevention" from the defender's viewpoint. The force inventories depicted in these figures may, by the nature of a steady state, refer to initial forces or to forces in being at any later time. A curve labeled  $R = \text{constant}$ , corresponding to a fixed spending ratio, represents the locus of opponent B's force compositions that lead to stalemate; i.e., no victory for either opponent. The solid curves may therefore be called "stalemate curves." Any point on such a curve represents a stalemate condition. Thus it is seen that stalemate does not result from a unique set of conditions, but from a one-parameter infinitude of conditions, representing different mixes of forces for prevention and cure.

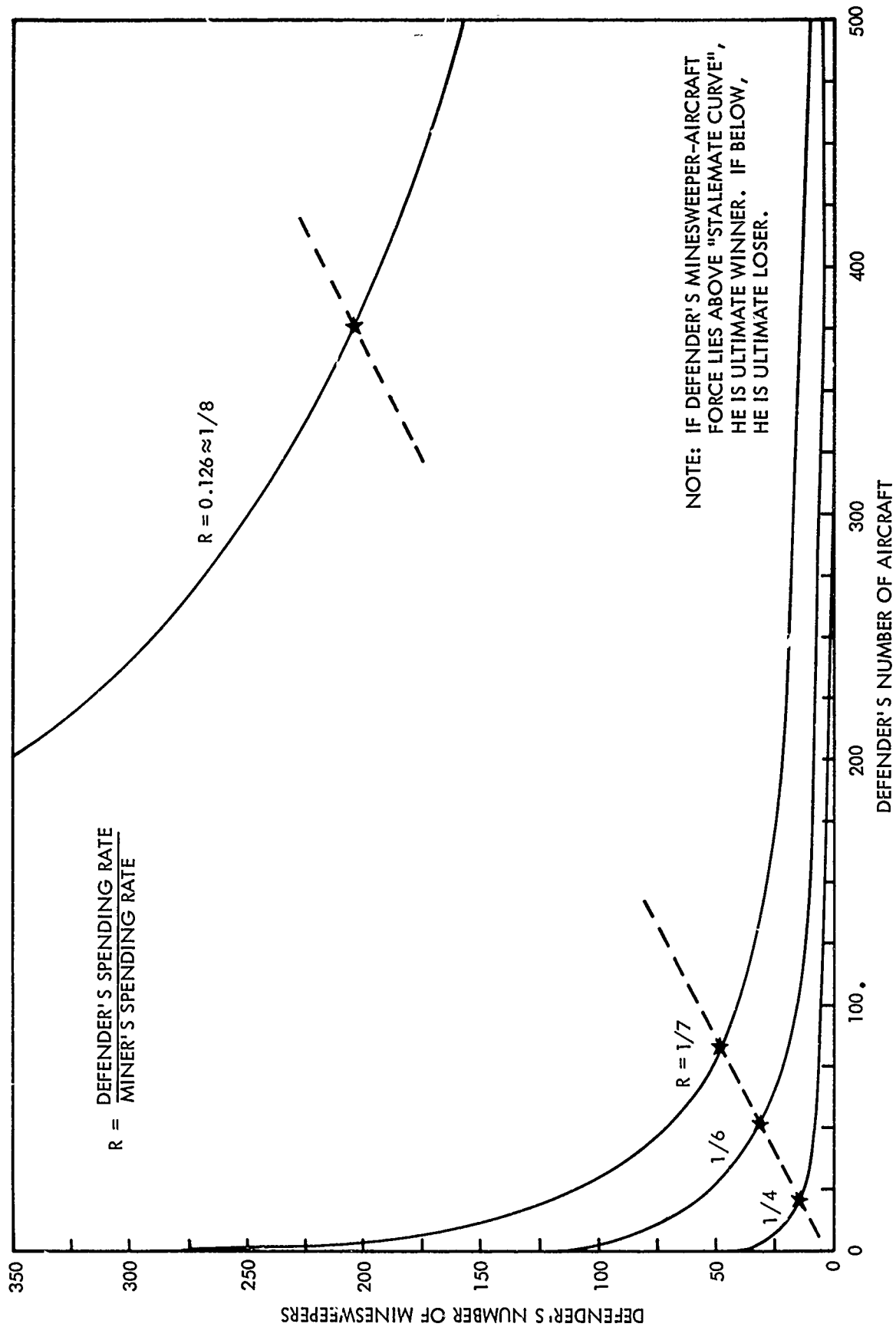


FIG. 1 STALEMATE CURVES FOR CASE OF GREATER SPENDING BY THE ATTACKER (MINER). HYPOTHETICAL PARAMETER VALUES. DEFENDER MAINTAINS CONSTANT SHIPPING LEVEL (25% PORT UTILIZATION).

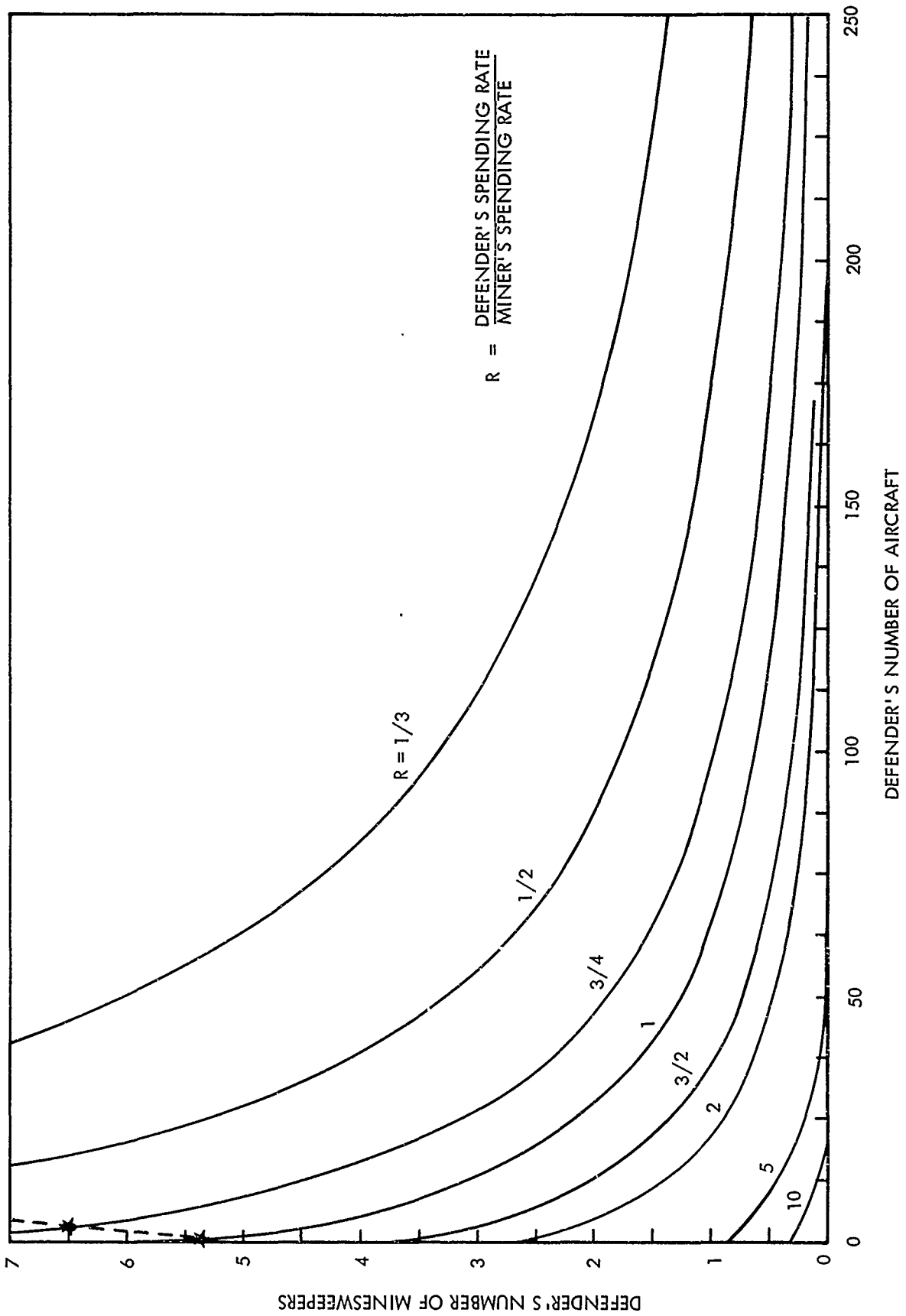


FIG. 2 STALEMATE CURVES FOR HIGHER VALUES OF SPENDING RATIO. HYPOTHETICAL PARAMETER VALUES. DEFENDER MAINTAINS CONSTANT SHIPPING LEVEL (25% PORT UTILIZATION).

13. A stalemate curve represents a sort of watershed. If the force actually available to opponent B, represented by a point on the cure vs. prevention diagram, is located in the region above the stalemate curve ( $R$  = appropriate constant), then the conflict will ultimately be won by opponent B. If the representative point lies below the stalemate curve the ultimate winner is A, provided -- as we here assume -- that opponent B's port utilization associated with the stalemate curve is at a minimum level to sustain his war effort. (Otherwise a lowering of the representative point below the stalemate curve generates not a win for A but simply a reduction of B's port utilization.) In general the rate of winning is greater the farther the representative point lies away from the stalemate curve. (This is all we can say about time relationships without solving the time-dependent equations.) The "representative point," as noted, could refer either to B's initial forces in being at the start of the war or to whatever he might survive with after some initial flurry of transient actions.

14. The reader must again be cautioned not to take seriously the specific curves given here. They are based for the most part on parameter values that bear little relation to reality. The curves are given merely to illustrate the type of information yielded by the model. With this in mind we can note certain qualitative features of the curves in Figures 1 and 2. The steep slopes of the stalemate curves near zero aircraft mean that to rely wholly on "cure" is uneconomical. One could achieve the same stalemate with much fewer minesweepers by investing in a few aircraft for "prevention." Similarly, the very long tails on the stalemate curves near zero minesweepers mean that overreliance on prevention is uneconomical. Evidently the best strategy for opponent B is to choose a mix of preventive and curative forces whose representative point lies somewhere between the extremes just mentioned. The best point is shown in Appendix B to be the "point of indifference," denoted by a star on the curves in the figures, at which the numerical value of the slope of the stalemate curve is equal to the negative of the ratio of aircraft replacement cost to minesweeper replacement cost (0.5 in our example). This point has the property that stalemate is maintained if a dollar is taken away from aircraft replacement and invested in minesweeper replacement, or vice versa. The loci of different "points of indifference" for different  $R$ -values is shown in Figures 1 and 2 as a dashed line. (The scale changes between these two figures make the dashed lines look different, but they represent the same thing.) The significance of the dashed line is this: If opponent B's force mix representative point lies above and to the left of the dashed line, B has invested too heavily in minesweepers (he could have obtained more protection for equal investment by buying relatively more aircraft); whereas, if the representative point lies below and to the right, B has invested too heavily in aircraft. The dashed curve lies so close to a straight

line of slope 0.5 through the origin that for this particular example we could formulate the simple rule of thumb: Opponent B should invest roughly equal total dollar amounts in minesweepers and in aircraft. For other examples presumably other rules would apply.

15. The significance of the limiting R-values,  $R = R_0 = 0.125$  and  $R = R_1 = 25.0$ , is the following: If the spending ratio  $R$  equals or exceeds  $R_1$  the defender is out-spending the attacker so heavily that the latter must lose. If  $R$  equals or is less than  $R_0$ , the attacker is out-spending the defender so heavily that the latter must lose, regardless of his initial inventory of aircraft and minesweepers. The approach to the latter condition is shown dramatically in Figure 1. If the attacker A outspends the defender B by 4:1, or 6:1, or 7:1 the corresponding stalemate curves lie close together in a region of reasonable force inventories for opponent B. But as the spending ratio approaches 8:1 ( $R = 0.125$ ) the stalemate curve suddenly moves out toward infinity. This strong nonlinearity of behavior as  $R \rightarrow R_0$  means that if opponent A can muster a spending ratio of, say, 7:1 he is foolish not to dig a little deeper and make it 8:1 or more -- for in this way he becomes unbeatable. A similar situation applies, with the roles of A and B reversed, near the opposite boundary,  $R \rightarrow R_1$ . This could in either case be called a landslide effect, by which the big spender swamps his opponent.

16. Figures of merit based on the stationary-state solutions of the Lanchester model are discussed in Appendix B. In attempting to deal with questions such as those listed in Section 3, above, it would be necessary to select a figure of merit from among candidates such as those discussed in Appendix B, and to examine the effect on it of parameter variations suggested by the question to be answered. For example, there is the question of what a mine should cost. By considering the effect on a figure of merit of successively cheaper mine designs, it should become apparent at what point too much effectiveness has been traded off. This point would not in general coincide with the point of maximum effectiveness/cost ratio, since here the entire campaign outcome is being considered, and "effectiveness" refers both to minesweeper targets and to other targets, and takes deliverability into account.



5. SUMMARY

17. We have seen that a good deal can be learned in a qualitative way about the nature and course of a mining campaign, merely by studying the steady-state solutions of a Lanchester model. To proceed further toward realism, or to seek specific implications for mine design and tactics, would require classified data inputs.

## APPENDIX A

Lanchester's Equations for Mine Warfare

A-1. Opponent A mines opponent B's ports in a sustained campaign aimed at attrition of B's "target" traffic. We shall attempt to describe the situation by Lanchester-type equations. Our objective is: initially, to gain a feeling for the dominant factors influencing the outcome of full-scale, opposed mining campaigns in general; ultimately, to be able to answer specific questions related to mine design and tactics for such warfare, as discussed in the text.

Notation

$A(t)$  = A's mine-delivery force active at time  $t$  (number of units).

$B_1(t)$  = B's mine countermeasures (sweeping) force active at time  $t$  (number of "average" minesweepers).

$B_2(t)$  = B's attrition forces active in preventing minelaying at time  $t$  (number of units).

$B_3(t)$  = B's shipping or other "target" (e.g., submarine) traffic force active at time  $t$  (number of operational vessels).

$A^*(t)$  = A's rate of buying (and deploying) new mine-delivery forces at time  $t$  (\$M/month).

$B_1^*(t)$  = B's rate of buying new mine countermeasures forces at time  $t$  (\$M/month).

$B_2^*(t)$  = B's rate of buying new attrition forces to prevent minelaying at time  $t$  (\$M/month).

$B_3^*(t)$  = B's rate of buying new shipping or other "target" units to replace losses at time  $t$  (\$M/month).

$T(t)$  = average level of mine threat to B's shipping or other "payoff" traffic at time  $t$  (mines assumed optimally deployed, also mine countermeasures, both proportionally to port "susceptibilities").

$B_T^*(t)$  = B's total force-buying rate at time  $t$ , expressed as a spending rate (\$M/month).

$A_T^*(t)$  = A's total force-buying rate, the sum of mine procurements plus delivery-force procurements at time  $t$ , expressed as a spending rate (\$M/month).

### Time-dependent Equations

A-2. The model simplifies by omitting operating costs and considering force procurement and expendable material costs only. We have

$$B_T^*(t) = B_1^*(t) + B_2^*(t) + B_3^*(t) \quad (1)$$

and

$$A_T^*(t) = A^*(t) + pA(t), \quad (2)$$

where  $p$  is a constant,  $pA(t)$  represents mine material costs proportional to available delivery forces (presumed to work steadily at renewing the minefields), and  $A^*(t)$  represents A's spending rate for procurement of new (replacement) delivery units. We have for the rate of threat change

$$\frac{dT}{dt} = aA(t) - bB_1(t) \cdot T(t) - cB_3(t) \cdot T(t) - qT(t), \quad (3)$$

where  $a, b, c, q$  are constants. Here  $aA(t)$  represents the rate of increase in threat level (number of mines present) due to planting by A's active units,  $bB_1(t) \cdot T(t)$  represents the decrease in threat

level due to B's sweeping, which is jointly proportional in effectiveness to how much there is to sweep (existing threat level) and how much force is available to sweep it. Similarly,  $cB_3(t) \cdot T(t)$  represents the effect of "sweeping" by B's ships that activate mines. The last term,  $-qT(t)$ , can represent a possible decay in the effectiveness of planted mines with time.

$$\frac{dA(t)}{dt} = e A^*(t) - f B_2(t) \cdot A(t), \quad (4)$$

where  $e, f$  are constants. This says that the increase of A's force per unit time is proportional to his rate of buying new forces, and the decrease is proportional jointly to the amount of attritional countermeasures forces  $B_2(t)$  mobilized by B and to the amount  $A(t)$  of A's minelaying forces exposed to the attrition.

$$\frac{dB_1(t)}{dt} = g B_1^*(t) - h T(t) \cdot B_1(t), \quad (5)$$

where  $g, h$  are constants. This says that B's mine countermeasures forces increase proportionally to his rate of procuring new forces of this type, and decrease proportionally to the threat level of the minefields they have to work in and to the number of minesweepers exposed to that threat. This supposes that mine countermeasures forces are subject to destruction by mines, but ignores any other military attacks to which they might be subject.

$$\frac{dB_3(t)}{dt} = j B_3^*(t) - k T(t) \cdot B_3(t), \quad (6)$$

where  $j, k$  are constants. This says that B's shipping forces grow proportionally to replacement rate and decrease proportionally to mine casualties, which are jointly proportional to threat level and shipping level.

$$\frac{dB_2(t)}{dt} = m B_2^*(t) - n A(t) \cdot B_2(t), \quad (7)$$

where  $m$ ,  $n$  are constants. This says that B's attritional forces directed at preventing minelaying grow proportionally to replacement rate and decrease proportionally to the size of the enemy force to be combatted (on the assumption that A's mine-delivery units elect to fight back) and to the number of B's attritional force units  $B_2(t)$  exposed to such counter-action.

### Steady State

A-3. If a steady state exists, all time derivatives are zero and quantities indicated as functions of time become constants. The mathematics simplifies accordingly. The seven equations become

$$B_T^* = B_1^* + B_2^* + B_3^*, \quad (1a)$$

$$A_T^* = A^* + pA, \quad (2a)$$

$$aA = bTB_1 + cTB_3 + qT, \quad (3a)$$

$$eA^* = fAB_2, \quad (4a)$$

$$gB_1^* = hTB_1, \quad (5a)$$

$$jB_3^* = kTB_3, \quad (6a)$$

$$mB_2^* = nAB_2. \quad (7a)$$

A-4. Let us treat the over-all economic investment rates on the two sides,  $A_T^*$  and  $B_T^*$ , as known, given constants. Then the above constitute seven equations in the nine unknown variables  $A$ ,  $A^*$ ,  $T$ ,  $B_1$ ,  $B_2$ ,  $B_3$ ,  $B_1^*$ ,  $B_2^*$ ,  $B_3^*$ . The system is underdetermined, so it is necessary to treat two more of these variables as given. We may choose  $B_1$  and  $B_2$  as an example.

From (2a) and (4a) we have

$$A_T^* = \frac{f}{e} B_2 A + p A,$$

or

$$A = \frac{A_T^*}{p + \frac{f}{e} B_2} \quad (8)$$

and

$$A^* = \frac{\frac{f}{e} B_2 A_T^*}{p + \frac{f}{e} B_2} \quad (9)$$

From (7a)

$$B_2^* = \frac{\frac{n}{m} B_2 A_T^*}{p + \frac{f}{e} B_2} \quad (10)$$

From (3a)

$$B_3 T = \frac{\frac{a}{c} A_T^*}{p + \frac{f}{e} B_2} - \frac{f}{c} B_1 T - \frac{q}{c} T,$$

From (1a) and (5a)

$$B_3^* = B_T^* - \frac{h}{g} B_1 T - \frac{\frac{n}{m} B_2 A_T^*}{p + \frac{f}{e} B_2}.$$

From (6a)

$$B_3^* = \frac{k}{j} B_3 T = \frac{\frac{ak}{cj} A_T^*}{p + \frac{f}{e} B_2} - \frac{bk}{cj} B_1 T - \frac{kg}{cj} T.$$

Equating these two expressions for  $B_3^*$ , we find

$$T = \frac{\left(\frac{ak}{cj} + \frac{n}{m} B_2\right) \left(\frac{A_T^*}{p + \frac{f}{e} B_2}\right) - B_T^*}{B_1 \left(\frac{bk}{cj} - \frac{h}{g}\right) + \frac{kg}{cj}}. \quad (11)$$

Hence from (5a)

$$B_1^* = \frac{\left(\frac{ak}{cj} + \frac{n}{m} B_2\right) \left(\frac{A_T^*}{p + \frac{f}{e} B_2}\right) - B_T^*}{\left(\frac{fkg}{cjh} - 1\right) + \frac{gkh}{cjh B_1}}, \quad (12)$$

and with  $\gamma \equiv bk B_1 / (bk B_1 + g)$ ,

$$B_3^* = \frac{\left(\frac{ah}{bg} \gamma + \frac{n}{m} B_2\right) \left(\frac{A_T^*}{p + \frac{f}{e} B_2}\right) - B_T^*}{\left(\frac{cjh}{fkg} \gamma - 1\right)}. \quad (13)$$

Consequently,

$$B_3 = B_1 \cdot \frac{f}{c} \left[ \frac{\frac{B_T^*}{A_T^*} \left(p + \frac{f}{e} B_2\right) \gamma^{-1} - \left(\frac{ah}{bg} + \gamma \frac{n}{m} B_2\right)}{\left(\frac{ak}{cj} + \frac{n}{m} B_2\right) - \frac{B_T^*}{A_T^*} \left(p + \frac{f}{e} B_2\right)} \right]. \quad (14)$$

If the quantities  $A$ ,  $A^*$ ,  $T$ ,  $B_1^*$ ,  $B_2^*$ ,  $B_3^*$ , given by equations (8)-(13), are nonnegative then the steady-state conditions (1a)-(7a) are identically satisfied, the time derivatives in equations (3)-(7) vanish at all times, and the quantities  $B_1$ ,  $B_2$  appearing in equations (8)-(13) are constants identifiable as initial values -- "initial," at any rate, if time is measured from the onset of a steady state, and we ignore any earlier transient effects. Also not treated are any one-shot transient expenditures, such as for ship degaussing.

A-5. It is convenient to rewrite Equations (1a)-(7a) or (8)-(14) in a form that simplifies the study of necessary conditions for a steady state. Let

$$\begin{aligned} R &\equiv \frac{B_T^*}{A_T^*}, & R_o &\equiv \frac{en}{f.m}, \\ R_1 &\equiv \frac{ak}{pcj}, & \epsilon_o &\equiv \frac{cjh}{bkq}, \\ K &\equiv \frac{f}{ep}, & \epsilon &\equiv \frac{\epsilon_o + B_1}{bB_1 + q}, \quad 0 \leq \epsilon \leq \epsilon_o. \end{aligned}$$

Then

$$A = \frac{(A_T^*/p)}{1 + KB_2}, \quad (15)$$

$$A^* = \frac{KB_2 A_T^*}{1 + KB_2}, \quad (16)$$



$$T = \frac{A_T^* \left[ (R_1 - R) - k B_2 (R - R_0) \right]}{B_1 \left( \frac{1}{\epsilon} - 1 \right) (1 + k B_2)}, \quad (17)$$

$$B_1^* = \frac{A_T^* \left[ (R_1 - R) - k B_2 (R - R_0) \right]}{\left( \frac{1}{\epsilon} - 1 \right) (1 + k B_2)}, \quad (18)$$

$$B_2^* = \frac{A_T^* B_2 (n/m p)}{1 + k B_2}, \quad (19)$$

$$B_3^* = \frac{A_T^* \left[ (R - \epsilon R_1) + k B_2 (R - R_0) \right]}{(1 - \epsilon) (1 + k B_2)}, \quad (20)$$

$$B_3 = \frac{(1 + B_1 \epsilon)}{\epsilon} \left[ \frac{(R - \epsilon R_1) + k B_2 (R - R_0)}{(R_1 - R) - k B_2 (R - R_0)} \right]. \quad (21)$$

A necessary condition for the existence of a steady state is that all quantities on the left-hand sides of equations (15)-(21) be nonnegative, and that none of the three quantities  $B_1^*$ ,  $B_2^*$ ,  $B_3^*$  exceed  $B_T^*$ .

These requirements imply restrictions on the allowed ranges of parameters. Consideration of realistic parameter values indicates that in general

$$R_1 > R_0. \quad (22)$$

Attention will be confined in the following discussion to the case in which condition (22) is satisfied.

A-6. In this case it is readily shown that for a steady state of warfare to exist in which neither opponent is the winner ( $B_3$  finite and nonvanishing) it is necessary that the ratio of resource investments,  $R = B_T^*/A_T^*$ , lie in the range

$$R_1 > R > R_0 \quad (23)$$

and that B's investment in attritional forces,  $B_2$ , lie in the range

$$\frac{(R_1 - R)}{K(R - R_0)} > B_2 > \frac{(\epsilon R_1 - R)}{K(R - R_0)} \text{ or } 0, \text{ whichever is greater.} \quad (24)$$

If conditions (23) and (24) are not satisfied, one or the other opponent wins, independently of initial force inventories.

A-7. For some purposes it may be more convenient to take  $B_1$ ,  $B_3$  as the independent variables (equal to their "initial values" at the onset of the steady state), instead of  $B_1$ ,  $B_2$ . In this case we find from equation (21) that

$$B_2 = \frac{1}{K(R - R_0)} \left[ \frac{(\epsilon_0 B_1 + c B_3) R_1}{(q + \epsilon B_1 + c B_3)} - R \right], \quad (21)'$$

and this can be used to eliminate  $B_2$  from equations (15) - (20). Similarly it may be desired to take  $B_2$ ,  $B_3$  as the independent variables.

In this case equation (21) yields

$$B_1 = \frac{cB_3[(R_1 - R) - KB_2(R - R_0)] - q[R + KB_2(R - R_0)]}{h[(R - \epsilon_0 R_1) + KB_2(R - R_0)]}, \quad (21)''$$

We then evaluate  $\epsilon$  by means of

$$\epsilon = \frac{(cj h / k q) B_1}{h B_1 + q}$$

and use this in equations (17), (18), and (20).

A-8. Suppose a fixed value of  $B_3$ , opponent B's target traffic level, is chosen equal to a threshold value, such that for smaller values of  $B_3$  it can be certified that opponent B is losing the war. (In the absence of a more comprehensive model, this choice of  $B_3$  [or equivalently of a minimum port utilization factor  $f_p$ ] must be a matter of judgment.) Secondly, choose a realistic value of spending ratio  $R$  of interest, and suppose that this characterizes the actual conflict. The values  $B_3$ ,  $R$  determine a stalemate curve  $C$  on the cure-vs.-prevention diagram. If the initial composition of opponent B's minesweeper and fighter aircraft forces in being at the start of hostilities is represented by a point on the diagram lying below curve  $C$ , the stalemate condition is not satisfied and opponent B's target traffic, being inadequately defended, falls below the threshold value  $B_3$ . In this case opponent A is assured of an eventual win. Similarly, if the point representing opponent B's initial forces lies above curve  $C$ , opponent B can either increase his target traffic level above the chosen value  $B_3$  or keep  $B_3$  constant and use his surplus resources to buy more minesweepers and aircraft than he is losing, thus adding progressively to his preponderance and assuring an eventual win. The stalemate curve thus, as stated in the text, defines a watershed for the winning of the conflict in terms of adequacy of B's initial force inventory. If on the other hand, conditions (23), (24) are not satisfied, so that a stalemate curve does not exist, the winner of the conflict is predetermined in favor of one opponent or the other, independently of initial forces in being. That is, one opponent dominates the other in terms of spending rate so thoroughly that even an indefinitely large initial force advantage to either would make no difference in the ultimate outcome. Although a true stalemate lacks practical interest we confine attention to this case here, because a description of the transient approach to steady state, or

of the rate of winning by either side, would require solution of the time-dependent equations, (1) - (7), which we do not attempt. In being content to learn who wins and under what conditions, we are not exploiting the full potentialities of Lanchester analysis for describing the time dimension in warfare. But we are presumably learning most of what it is useful to know and are gaining the whole-war bird's-eye-view for which such analysis is by its nature best adapted. The uncertainties of actual warfare are so extreme that any theoretical model can at best serve as a conditioner of expectations.

### Parameters

A-9. Use of equations (15) - (20) and, say, (21)" to determine the outcome of a mining campaign requires the specification of four numbers, which may be designated "campaign variables"; viz.,

- a.  $A_T^*$ , the total spending rate by the miner, opponent A.
- b.  $B_T^*$ , the total spending rate by the victim of mining, opponent B.
- c.  $B_3$ , the shipping level maintained by opponent B during the steady-state phase of the campaign, or the "initial value" of same.
- d.  $B_2$ , the steady-state (or "initial value") anti-mining attritional force level (i.e., forces that directly attack the mine-laying units) of opponent B.

A-10. It also requires evaluation of the weaponry and cost parameters  $p, a, b, c, e, f, g, h, j, k, m, n, q$ . The latter evaluations will now be discussed.

### Parameters $e, q, j, m$

A-11. These are cost parameters, defined as follows:

- $e$  = reciprocal of procurement cost for average mine-delivery unit  $[(\$M)^{-1}]$ .
- $g$  = reciprocal of procurement cost for average minesweeper  $[(\$M)^{-1}]$ .
- $j$  = reciprocal of procurement cost for average target vessel (merchant ship or submarine)  $[(\$M)^{-1}]$ .
- $m$  = reciprocal of procurement cost for average anti-minelayer attrition unit plus pro-rated cost of detection and other support gear  $[(\$M)^{-1}]$ .

The analysis is too crude to warrant any concern about differences between averages of reciprocals and reciprocals of averages -- particularly since in the present case such problems are artifacts that could be eliminated by redefining the unstarred quantities,  $A(t)$ ,  $B_1(t)$ , etc. in terms of dollar investment levels instead of force levels. This would make  $e$ ,  $g$ ,  $j$ ,  $m$  dimensionless and would permit them to be defined equal to unity.

#### Parameter p

A-12. The parameter  $p$  (equation (2)) represents opponent A's mine cost per active mine-delivery unit per unit time. Delivery and operating costs could be considered as well (likewise for parameters  $e$ ,  $g$ ,  $j$ ,  $m$ ), but we have here arbitrarily excluded operating costs on both sides, as if they came "out of another pocket," on the philosophy that a war is on, both sides are going to operate whatever forces they procure, and the real limitations are on material procurement and fabrication capacity. We suppose that wartime difficulty of procurement is proportionally related to peacetime procurement cost. This is obviously questionable. It is likely that many "cost-effectiveness" considerations are valid only in peacetime and have little to do with wartime scarcity-controlled "costs." For present purposes a crude treatment based on peacetime costs should suffice to guide peacetime decisions. Without detailed predictions of wartime scarcities and development of a "new economics" to describe their implications, one could not pretend to accuracy. Even then, uncertainty as to the nature of the war and the stakes involved makes all quantifications speculative.

A-13. In view of equation (27), below, the parameter  $p$  may be evaluated as

$$p = n_s n_m c_m, \quad (26)$$

where  $n_s$ ,  $n_m$  are defined below and

$c_m$  = procurement cost per mine in quantity production (\$M/mine).

#### Parameter a

A-14. This is one of the critical parameters, since it measures the effectiveness of A's forces in building up a mine threat. To evaluate it and some of the other parameters explicitly, one needs to examine in detail a specific scenario. This will be done below. First, we need a definition of "threat"  $T(t)$ . Ordinarily, threat is

defined as a probability, but we find it more convenient here to define it in terms of expectation, and to give it the dimensions of inverse miles of ship track through the minefield. (This is arbitrary but convenient.) That is,

$T$  = Expected number of sinking-degree mine attacks on an "average target vessel" per nautical mile of target track through the minefield  $[(n. \text{ miles})^{-1}]$ .

A-15. The choice of average target vessel would be influenced by scenario. For a shipping blockade it might be an "average merchant ship," for an ASW campaign it might be a notional "average" enemy submarine, etc. Let

$N$  = total number of mines delivered per month by opponent A.

$n_s$  = average number of sorties per mine-delivery unit per month.

$n_m$  = average number of mines carried per sortie.

Then

$$N = n_s n_m A, \quad (27)$$

where  $A$  is the previously-defined  $A(t)$  with time-dependence suppressed. Further, assume that opponent B's traffic is channelized and let

$A_m$  = total shipping channel area mined (sq. n. mi.) = (total channel length for all ports)  $\times$  (mean channel width).

$F$  = average fraction of delivered mines planted in shipping channels.

$f_1$  = average fraction of mines in shipping channels operable and on count 1 at any time (includes mine reliability factor).

$w$  = mean actuation width of a mine, averaged for targets and mine influences employed (n. mi.). (It is assumed that  $w$  does not exceed mean channel width.)

$f_2$  = average fraction of mine explosions effective in producing damage equivalent to sinking.

Then the product

$$\frac{N_w f_1 f_2 F}{A_m} = \frac{n_s n_m f_1 f_2 F w}{A_m} \cdot A$$

represents the expected number of lethal attacks by mines laid in one month per mile of "average target vessel" track through the mine-field. Considering the threat definition given above, we see that this quantity is equal to the contribution to  $(dT/dt)$  [all times are measured in months] from opponent A's total minelaying activities; that is, it is equal to  $aA$  (see equation (3)). Consequently, the parameter  $a$  is evaluated as

$$a = \frac{n_s n_m f_1 f_2 F w}{A_m} (\text{miles} \times \text{months})^{-1} \quad (28)$$

#### Parameter b

A-16. This parameter is also of critical importance, since it measures sweeper effectiveness. If we define  $B_1$  as the mean number of notional "average" minesweepers active at any time, consider influence mines with ship counters, and let

$W_s$  = notional average minesweeper swept path width, averaged for mine influences in the mine mix actually employed (n. mi.). (It is assumed that  $W_s$  does not exceed mean channel width.)

$v_s$  = average minesweeper speed when sweeping (kts. x hrs./month).

$n_c$  = av. no. of ship counts run off per sweeper pass on an influence mine located within sweep path ( $1 \leq n_c \leq f_1^{-1}$ ).

$d_s$  = sweeper average duty cycle (fraction of the month spent in sweeping),

then it is seen that parameter b in equation (3) has the value

$$b = \frac{W_s n_s d_s n_c}{A_m} (\text{month})^{-1}. \quad (29)$$

It is here assumed that sweeping is of the influence type, so that mines not on counts 1 to  $n_c$  are not swept on a pass. (If some kind of hunting or sweeping, e.g., mechanical, is employed that eliminates all mines present within width  $W_s$  on each sweep, the above expression for b must be multiplied by  $f_1^{-1} \cdot n_c^{-1}$ .) The duty cycle  $d_s$  is affected by the opponent's tactics in using or not using "obstructors" to interfere with sweeping efforts. By developing a parametric expression for  $d_s$  as a function of obstructor effort, and doing the same for F (to reflect division of effort between minelaying and obstructor laying), it should be possible to use the present model to deduce optimum obstructor tactics. This will not be done here, but it is mentioned as an example of extended applications of the model.

#### Parameter c

A-17. We define  $B_3$  as the mean number of "target" traffic vessels operable (not sunk nor seriously damaged) at any time. These target vessels can be either surface ships or submarines. Let

$v_t$  = target vessel mean speed when traversing minefield (knots x hrs./mo.)

$d_t$  = target vessel average minefield duty cycle (fraction of vessel's time spent in traversing minefield).

By analogy with equation (29) we have

$$c = \frac{W v_t d_t}{A_m} (\text{month})^{-1}. \quad (30)$$



Let

$T_1$  = mean target vessel cycle time from entering port to next entering port (months), both ports belonging to opponent B.

$B_c$  = mean swept channel width (n. mi.).

$n_o$  = number of ports mined.

$T_t$  = average target vessel one-way minefield transit time when entering or leaving port (months).

The average channel length per port is  $(A_m/n_o B_c)$  and is also equal to the product  $v_t T_t$ ; so

$$T_t = \frac{A_m}{n_o B_c v_t}$$

Since there are two minefield transits per port call,

$$d_t = \frac{2 T_t}{T_1} = \frac{2 A_m}{n_o B_c v_t T_1} \quad (31)$$

Hence equation (30) may alternatively be written as

$$c = \frac{2 w}{n_o B_c T_1} \quad (\text{month}^{-1}) \quad (30)'$$

The ship cycle time  $T_1$  may be effectively increased by down-times for maintenance or other effects of less than total ship availability.

Port Utilization

A-18. Let

$f_p$  = port utilization fraction (average fraction of port capacity in use) for "target" class vessels.

$P$  = total tonnage handling capacity of all ports (GWT per month) for target class vessels.

$T_o$  = average tonnage of a target class vessel (GWT).

The average number of target class vessels per month calling at all ports may be written either as  $(B_3/T_1)$  or as  $(f_p P/T_o)$ . Consequently  $f_p$  may be expressed in terms of  $B_3$  as

$$f_p = \frac{T_o B_3}{P T_1} \quad (32)$$

$B_3$  being determined in terms of  $B_1$  and  $B_2$  by equation (21). The utilization factor  $f_p$  provides an alternative measure of mine warfare effectiveness. If on the other hand  $f_p$  is known,  $B_3$  is determined,

$$B_3 = \frac{f_p P T_1}{T_o} \quad (32)'$$

and  $B_1$  is determined in terms of  $B_3$  and  $B_2$  by equation (21)". This was the assumption used in the text for deriving the stalemate curves of Figures 1 and 2, where  $f_p$  was assumed to be the least value required to sustain the war effort of opponent B.

Parameter  $\hat{f}$ 

A-19. This is a weapon effectiveness parameter of the classical Lanchester type. Opponent A's minelaying units, dispatched in small groups, are subject to a certain probability of being detected and met by some of B's attrition units. The conflicts actually occur in small "dogfights" of highly individual character, but we idealize by

considering the over-all force availabilities on the two sides,  $A(t)$  and  $B_2(t)$ , to be simply related to the forces present in these multiple individual conflicts. For the sake of being specific, let us consider aerial minelaying by opponent A, countered by fighter aircraft attacks by B. Let

$n_a$  = average number of simultaneous sorties per mining mission flown by opponent A against a particular port (raid size).

$p_{AB}$  = probability per sortie that one of A's aircraft is destroyed by one of B's aircraft available to attack it.

Then  $-(dA/dt)$ , opponent A's monthly aircraft loss rate, may be expressed either as  $fB_2A$  (equation (4)), or as the product of  $n_sA$ , opponent A's total number of aircraft sorties per month, times  $p_{AB}$ , the probability of loss per sortie per B-aircraft available to attack the A-aircraft, times  $B_2$ , the total number of B-aircraft, reduced by the factor  $(1/n_o)$ , which expresses the fact that A's sortie is directed only against a particular port (B's aircraft based near other ports are assumed to be too remote to contribute to the fight; otherwise a reduced effective value  $n'_o$  equal to the number of widely-separated ports, could be substituted for  $n_o$ ), and further reduced by the factor  $(1/n_a)$ , which expresses the fact that the  $(B_2/n_o)$  aircraft of opponent B dispatched on the average to deal with any particular minelaying mission near a given port must divide their fire among  $n_a$  of A's aircraft, so that the average number of B's aircraft available to attack any particular one of A's aircraft is  $(B_2/n_on_a)$ . Equating these alternative expressions for A's monthly loss rate, we obtain

$$f = \frac{n_s p_{AB}}{n_o n_a} \quad (33)$$

A-20. Here, in considering active aircraft forces only, we ignore opponent B's need to keep a tactical reserve of anti-mining aircraft, just as we ignore the existence of pools of "dud" aircraft on both sides, support activities, etc. The model would accommodate other assumptions. From equation (33) it is obvious that attrition rates depend on tactics, and that massed attacks (preferably with surprise, so that B cannot shift his defenses) are most effective. The present treatment is readily modified to describe the effect of fighter escort on minelaying aircraft losses.

A-21. Further development of the model beyond this stage might be fruitful. For example: Opponent A would like to reduce his attrition coefficient  $f$  by making  $n_a$  as large as possible. What limits  $n_a$ , besides logistic expediency and tactical habits? Clearly, it is the required frequency of mining re-attacks on a given port. The lower this frequency the larger the average raid size that can be achieved, given a limited total number of flying hours available per month. What controls the required frequency of re-attacks? Clearly, the mine characteristics or settings -- mean time to arm, maximum ship counts, etc. Thus the model, exploited in greater depth, could seemingly tell us how certain detailed mine design features affect the over-all course of the campaign. Possibilities of this kind make the analysis seem worth pursuing, even in the face of obvious difficulties of quantifying some factors.

#### Parameter n

A-22. Let

$P_{BA}$  = probability per sortie that one of A's aircraft destroys one of B's aircraft that attempts to attack it.

By a derivation similar to that given for the parameter  $f$ , we obtain

$$n = \frac{n_s p_{BA}}{n_o n_a} \quad (34)$$

It is tacitly assumed that dogfights involve separate, individual passes by B's aircraft against A's, and that the principal hazard to an attacking aircraft is from the aircraft attacked. Other assumptions would lead to modified evaluations of  $n$ . Note that  $f$  and  $n$  are unequal if the fighting capabilities of the aircraft differ,  $P_{AB} \neq P_{BA}$ .

#### Parameter h

A-23. This measures the vulnerability of minesweepers to mines. Obviously,  $h$  is sensitive to mine design and minelaying strategy. To the extent that sophisticated mines can sense the difference between sweepers and target vessels, the miner can elect either to attack minelayers or to let them pass and attack only targets. Not much seems to be known about the correct choice of strategy in such a case. The present model could seemingly be used to study the question, but this

will not be attempted here. Let

$k_1$  = vulnerability of average minesweeper to sinking by mines relative to vulnerability of "average target vessel" (= average number of sinkings of sweepers per sinking of target vessel for equal track mileages in minefield) [dimensionless ratio].

Then it is easily seen that

$$h = k_1 v_s d_s \quad (\text{n. mi./sweeper month}). \quad (35)$$

#### Parameter k

A-24. In this instance we are concerned with the attrition of "average target vessels" produced by the minefield. We have

$$k = v_t d_t, \quad (\text{n. mi./target vessel}) \quad (36)$$

and  $d_t$  may be further evaluated from equation (31) to yield

$$k = \frac{2 A_m}{n_o B_c T_1} \quad (36)''$$

#### Parameter q

A-25. The parameter  $q$  measures mine random loss rate from causes other than ship-count actuation. One contribution to  $q$  can be expressed as  $1/\tau_1$ , where  $\tau_1$  is the mean lifetime of the mine for deterioration after planting, in the absence of signals that would actuate it. Another contribution might be  $1/\tau_2$ , where  $\tau_2$  is a mean time after planting at which the mine is deliberately set to sterilize itself. If the mine is provided with a random arming delay, such that the probability of its becoming active per unit time is  $p_o$ , then in the limit of heavy sweeping counter-measures effort the mine will always be actuated soon after arming, in a time interval short compared to the mean arming delay,  $\tau_o$ , so that  $p_o \tau_o$  (month<sup>-1</sup>) approximates the

loss rate of a delayed-arming mine. The total loss rate is the sum of whatever contributions apply,

$$q = \frac{1}{\tau_0} + \frac{1}{\tau_1} + \frac{1}{\tau_2} + \dots \quad (37)$$

It is seen that arming delays affect the durability of a mining threat quite differently from the way ship counts do. The latter introduce a dependence on the level of traffic and countermeasures activity, while the former do not.

## APPENDIX B

Figures of Merit

B-1. A figure of merit is a single number, a function of all parameters of the problem, that increases steadily as the situation described by the model becomes more favorable to one opponent and decreases as it becomes more favorable to the other. Figures of merit furnish shortcuts for determining whether a given change in input parameters is good or bad from a particular opponent's viewpoint. By studying the change in a single number, one gets information about which opponent gains from situational changes, technical innovations, etc. The quantitative question of how much is gained is often less important than the qualitative question of whether any gain is made at all -- since the latter is all that need be known to accomplish optimizations.

B-2. The Lanchester model of Appendix A offers a number of possible figures of merit. Which is preferred may depend on the availability of input data. For example, if opponent B's initial minesweeping forces  $B_1$  and minelaying countermeasures forces  $B_2$  in being were known, or could be estimated, this would determine a point P on a cure-versus-prevention diagram such as Figure 1. An appropriate figure of merit for opponent A's posture would then be R, the ratio of B's total spending to A's total spending for a given value of port utilization factor,  $f_p$ , appropriate to the stalemate curve passing through P. This meets our description of a figure of merit because an increase in R, reflecting an increase in opponent B's relative spending effort required to achieve stalemate, is always favorable to opponent A.

B-3. To evaluate R we may solve for it in equation (21), treating  $B_1$ ,  $B_2$ ,  $B_3$  as known to obtain

$$R = \frac{\epsilon_0 R_1 B_1 + K R_0 B_2 (q + \epsilon B_1) + c B_3 (R_1 + K R_0 B_2)}{(1 + K B_2)(f + \epsilon B_1 + c B_3)} \quad (38)$$

We observe that if  $q = 0$  then  $B_1$  and  $B_3$  enter only through their ratio. This means that the target traffic level opponent B can maintain is directly proportional to the size of his minesweeper force, provided the loss of mines by non-ship-count means is negligible.

B-4. If opponent B's initial posture (location of point P) is unknown, as might be true for purposes of long-term future planning, the following alternative figure of merit may be of interest. First, choose a value of spending ratio R of interest, perhaps on the basis of "conservative offensive planning." Then choose a minimum value of port utilization  $f_p$  that would permit opponent B to stay in the fight. In the case of an anti-merchant shipping campaign  $f_p$  would correspond to the least flow of surface shipping required to maintain opponent B's war economy. In the case of an ASW mining campaign  $f_p$  would represent the threshold (if such exists) below which opponent B's operational submarine fleet would cease to be an unmanageable threat to opponent A's shipping. Then define a quantity

$$Q = \frac{B_1}{g} + \frac{B_2}{m},$$

where  $B_1$ ,  $B_2$  correspond to any point on the cure-versus-prevention diagram lying on the stalemate curve appropriate to the given values of R and  $f_p$ . The term  $B_1/g$  represents the total procurement cost of opponent B's initial inventory of minesweepers;  $B_2/m$  represents total cost of his initial inventory of counter-minelaying units (fighter aircraft). The quantity Q is of the nature of a figure of merit for opponent A's posture, because it represents the total cost of B's initial force inventory required to achieve stalemate. This increases with any situational change favoring A.

B-5. Since  $B_1$ ,  $B_2$  as just discussed correspond to any point lying on some particular stalemate curve, the value of Q is indeterminate; i.e., Q does not represent a single number. This is easily corrected. From opponent A's viewpoint it is conservative to suppose that opponent B optimizes his initial posture by choosing  $B_1$ ,  $B_2$  to minimize Q. Differentiating with respect to  $B_2$ , subject to the constraint of remaining on a stalemate curve ( $R = \text{constant}$ ,  $B_3 = \text{constant}$ ), we obtain

$$\frac{dQ}{dB_2} = 0 = \frac{1}{m} + \frac{1}{g} \left( \frac{dB_1}{dB_2} \right)_{\text{stalemate curve}}$$

or

$$\begin{aligned} \left( \frac{dB_1}{dB_2} \right)_{\text{stalemate curve}} &= -\frac{g}{m} = -\frac{\text{Fighter A/C unit cost}}{\text{Minesweeper unit cost}} \\ &= \text{Slope of stalemate curve.} \end{aligned} \quad (39)$$



This establishes the result stated in the text about B's optimum initial force composition. Equation (39) permits us to solve a simple quadratic (after differentiating equation (21)) for the coordinates  $B_1^+$ ,  $B_2^+$  of the starred "point of indifference" on curves such as those of Figures 1, 2. (For curves unmarked with a star, take  $B_2^+ = 0$ .) Then

$$Q^+ = \frac{B_1^+}{g} + \frac{B_2^+}{m} \quad (40)$$

is a single number satisfying all requirements for a figure of merit. It represents the least possible cost to opponent B to acquire an initial force inventory just sufficient to stalemate the campaign, given  $R$  and  $f_p$ . Obviously the bigger this is the better from A's point of view.

B-6. By following the procedure just outlined, we find an explicit expression for  $Q^+$  in terms of  $R$  and  $B_3$ , viz.,

$$Q^+ = \frac{1}{fmK(R-R_0)} \left[ f(\epsilon_0 R, -R) - \frac{mK}{g} (g + cB_3)(R-R_0) \right. \\ \left. + 2\sqrt{\frac{fmK}{g} (R-R_0)R, [cB_3(1-\epsilon_0) - g\epsilon_0]} \right]. \quad (41)$$

This is valid provided the right-hand side of the equation is real and non-negative. In general this means  $R$  small enough (but  $R > R_0$ ) and

$$B_3 \geq \frac{g\epsilon_0}{c(1-\epsilon_0)}.$$

The interpretation of  $Q^+$  is that it represents the minimum initial force investment by the defender to achieve stalemate, given fixed values of spending rate ratio  $R$  and shipping level  $B_3$ . The attacker wishes to design his mines and choose tactics to maximize  $Q^+$ .

B-7. An alternative is to treat the defender's initial investment  $Q$  and shipping level  $B_3$  as given and to find an optimum

(minimum) value  $R^+$  of spending rate ratio from the defender's viewpoint. This means expressing  $R$  from equation (38) as a function of  $Q$ ,  $B_2$ , and  $B_3$  and finding the optimum value of  $B_2$ . We omit this calculation here, since the method is clear.

B-8. The foregoing examples of figures of merit illustrate that the steady state solutions of a Lanchester model provide serviceable guides for optimizing mine warfare postures, even though neither opponent will in general choose stalemate as his war aim.

DISTRIBUTION

Copies

Office of Chief of Naval Operations  
Department of the Navy  
Washington, D. C. 20350  
(L. E. Hoisington (OP-323))

Chief of Naval Research  
Department of the Navy  
Washington, D. C. 20360  
(Code 463)

Chief of Naval Material  
Department of the Navy  
Washington, D. C. 20360  
(MAT-01)  
(MAT-03)

Commander  
Naval Ordnance Systems Command  
Washington, D. C. 20360  
(Library)  
(ORD-03)  
(ORD-05)  
(ORD-035)  
(ORD-05121)

2

Director  
Center for Naval Analyses  
1401 Wilson Blvd.  
Arlington, Va. 22209  
(Chief Scientist)

Superintendent  
Naval Postgraduate School  
Monterey, Calif. 93940

Commanding Officer  
U. S. Naval Schools,  
Mine Warfare  
U. S. Naval Base  
Charleston, S. C. 29408

Officer-in-Charge  
U. S. Naval Mine Engineering Facility  
U. S. Naval Weapon Station  
Yorktown, Va. 23491

DISTRIBUTION (CONT.)

Copies

20

Defense Documentation Center  
Cameron Station  
Alexandria, Va. 22314

Commander  
U. S. Naval Weapons Laboratory  
Dahlgren, Va. 22448

Commanding Officer  
U. S. Navy Mine Defense Laboratory  
Panama City, Fla. 32402

Systems Analysis Office  
U. S. Naval Ordnance Laboratory  
White Oak  
Silver Spring, Md. 20910  
(D. Q. Brumbaugh)  
(Dr. A. D. Solem)  
(C. B. Brown)

Naval Electronics Laboratory Center  
271 Catalina Boulevard  
San Diego, California 92152

Navy Underwater Sound Laboratory  
Fort Trumbull  
New London, Connecticut 06320

Naval Ship Research and Development Center  
Washington, D. C. 20007

Naval Undersea Research & Development Center  
San Diego, California 92152

Naval Underwater Weapons Research & Engineering Station  
Newport, Rhode Island 02844

Applied Physics Laboratory  
University of Washington  
Seattle, Washington 98105

Ordnance Research Laboratory  
State College, Pennsylvania 16801

Naval Air Development Center  
Johnsville, Warminister, Pennsylvania 18974

DISTRIBUTION (CONT.)

Copies

Woods Hole Oceanographic Institution  
Woods Hole, Massachusetts 02543

Scripps Institution of Oceanography  
San Diego, California 92152

Naval Oceanographic Office  
Washington, D. C. 20390  
(Code 037-B)

Naval Ship Research & Development Laboratory  
Panama City, Florida 32402

UNCLASSIFIED

Security Classification

| DOCUMENT CONTROL DATA - R & D   |   |                                    |
|---|---|------------------------------------|
| <i>(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)</i>   |   |                                    |
| 1. ORIGINATING ACTIVITY (Corporate author)  |   | 2a. REPORT SECURITY CLASSIFICATION |
| U. S. Naval Ordnance Laboratory<br>Silver Spring, Maryland  |   | UNCLASSIFIED                       |
|   |   | 2b. GROUP                          |
|   |   | --                                 |
| 3. REPORT TITLE   |   |                                    |
| APPLICATION OF LANCHESTER ANALYSIS TO A MINING CAMPAIGN   |   |                                    |
| 4. DESCRIPTIVE NOTES (Type of report and inclusive dates)   |   |                                    |
| --  |   |                                    |
| 5. AUTHOR(S) (First name, middle initial, last name)  |   |                                    |
| T. E. Phipps  |   |                                    |
| 6. REPORT DATE  | 7a. TOTAL NO. OF PAGES  | 7b. NO. OF REFS                    |
| 15 November 1969  | 41  | 2                                  |
| 8a. CONTRACT OR GRANT NO.   | 9a. ORIGINATOR'S REPORT NUMBER(S)   |                                    |
| b. PROJECT NO.  | NOLTR 69-204  |                                    |
| c. ORD-531-215/UF17-351-503   | 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) |                                    |
| d.  | --  |                                    |
| 10. DISTRIBUTION STATEMENT  |   |                                    |
| Each transmittal of this document outside the Department of Defense must have prior approval of NOL   |   |                                    |
| 11. SUPPLEMENTARY NOTES   | 12. SPONSORING MILITARY ACTIVITY  |                                    |
| --  | Naval Ordnance Systems Command  |                                    |
| 13. ABSTRACT  |   |                                    |
| <p>Lanchester-type equations are derived for a mine warfare campaign in which one opponent seeks to blockade the ports of another. The effects of mutual attrition of minelaying units, counter-minelaying forces, minesweepers, and target traffic (merchant shipping or submarines) are treated and conditions for victory or stalemate are derived. Only steady-state solutions of the equations are examined. These appear adequate to answer a broad range of questions about the campaign, including the effect of mine design and tactics on total investment levels needed to win. The particular advantage of this type of model is that it provides an easy, though crude, means of estimating the effect of even the most minor design and situational variables on the over-all course of the campaign. For this reason it is hoped that it can prove useful to the mine designer or tactician.</p> |   |                                    |

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801

UNCLASSIFIED

Security Classification

| 14. KEY WORDS                                   | LINK A |    | LINK B |    | LINK C |    |
|---|--------|----|--------|----|--------|----|
|   | ROLE   | WT | ROLE   | WT | ROLE   | WT |
| Analysis<br>Minefield<br>Blockade<br>ASW Mining |        |    |        |    |        |    |